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# THE PLASTIC BUCKLING OF LONG CYLINDRICAL SHELLS UNDER PURE BENDING†

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**Abstract**—The effect of nonlinear material behavior on the buckling of an infinitely long cylindrical shell under pure bending is studied. The maximum moment and associated curvature is determined as a function of material and geometric parameters. The curvature at which short wave-length bifurcations occur is also determined. The results are compared with behavior in the elastic range.

## INTRODUCTION

The buckling of elastic cylindrical shells in pure bending has been approached in several ways. When the shells are not very long, prebuckling ovalization of the cross section of the shell may be neglected. Seide and Weingarten[1] used a linear membrane prebuckling state in their analysis of elastic bifurcation buckling. Their results demonstrated that the maximum compressive stress in the shell at bifurcation was only slightly greater than the stress at bifurcation of the same shell in uniform axial compression. This stress is given by

$$\sigma_c^e = \frac{Et}{[3(1-\nu^2)]^{1/2}R} \tag{1}$$

where $E$ is Young's modulus, $\nu$ is Poisson's ratio, $t$ is the shell wall thickness and $R$ is the radius of the middle surface of the undeformed shell. The applied moment at bifurcation is thus slightly greater than

$$M_c^e = \frac{E\pi Rt^2}{[3(1-\nu^2)]^{1/2}}. \tag{2}$$

Brazier [2], in studying an infinitely long elastic shell, did take ovalization into account, but excluded the possibility of bifurcation. He showed that there existed a maximum moment to which the shell could be loaded, given by

$$M_\mu^e = 0.5443\, M_c^e. \tag{3}$$

The moment occurs when the overall curvature reaches

$$\kappa_\mu^e = 0.8165\, \kappa_c^e \tag{4}$$

where $\kappa_c^e$ is the overall curvature associated with the moment given by (2). This curvature is

$$\kappa_c^e = \frac{t}{[3(1-\nu^2)]^{1/2}R^2}. \tag{5}$$

The maximum inward deflection occurs at the points of maximum tensile and compressive stress. At collapse, this deflection is

$$\delta_\mu^e = \frac{2}{9}R. \tag{6}$$

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and the magnitude of the stress at these points is

$$\tau^e_\mu = 0.635\,\sigma^e_c. \tag{7}$$

Bifurcation from this non-linear state was considered by Akselrad[11], who produced an approximate solution for cylinders of variable length where both geometric non-linearities and edge constraints are taken into account, and then assumed that the cylinder would bifurcate into a short wave-length pattern similar to that of cylindrical shells in uniform axial compression. For short shells, the maximum moment obtainable approaches the results of Seide and Weingarten; and for very long shells it approaches the results of Brazier. A more rigorous approach by Almroth and Starnes[12] showed these results to be reasonably accurate. They introduce a small imperfection and use a large scale two-dimensional shell program to calculate the maximum moment as a function of the length to radius ratio of the shell.

Recently, a paper by Fabian[3] took geometric nonlinearities into account to determine the limit moment of the long elastic cylinder. Bifurcation from the nonlinear state was also investigated, and found to occur at loads slightly less than those of the limit state.

An early, approximate analysis of the same basic problem considered here which incorporated geometric and material nonlinearities into Brazier's problem was performed by Ades[4], who further assumed that the cross section of an infinitely long circular cylinder deforms into an ellipse.

The purpose of this paper is to extend the results of Brazier into the plastic range. In addition, bifurcation in the plastic range from the ovalized state will be investigated, and contact will be made with the elastic results of Fabian.

## GEOMETRY AND STRAIN-DISPLACEMENT RELATIONS

A point on a cylindrical surface of radius $R$ may be specified by the Cartesian axial and circumferential coordinates $x$ and $y$ on this surface. Together with the outward normal, these coordinates form a right-handed system. The coordinate that specifies the radial distance from the middle surface of the shell is $z$. The quantities $u$, $v$ and $w$ are the axial, circumferential and radial displacements from the undeformed cylindrical geometry when the shell is subject to loading. The thickness of the shell is represented by $t$.

Figure 1 shows schematically the deformed shape of an infinitely long circular cylindrical shell in pure bending. The upper figure shows the axial deformation. From symmetry, it is required that plane sections remain plane for an infinitely long shell in pure bending. It follows from this that the longitudinal stretching strain is

$$E_{xx} = -\kappa h \tag{8}$$

where $\kappa$ is the overall curvature and $h$ is the distance of the deformed middle surface from the neutral axis. With $v$ and $w$ as the displacements in the plane of the representative cross-section shown, and with $y = 0$ taken as the fiber of maximum axial compression, $h$ is then

$$h = (R + w)\cos(y/R) - v\sin(y/R). \tag{9}$$

From Brazier's study it is known that the in-plane displacements are on the order of $R/10$ at maximum moment; thus, the strain-displacement relations of nonlinear ring theory will be used to calculate the in-plane stretching and bending strains. If $e$ and $\beta$ are defined as

$$e = \frac{dv}{dy} + \frac{w}{R} \tag{10}$$

$$\beta = \frac{v}{R} - \frac{dw}{dy} \tag{11}$$

then the circumferential stretching strain is

$$\eta = e + \frac{1}{2}e^2 + \frac{1}{2}\beta^2. \tag{12}$$

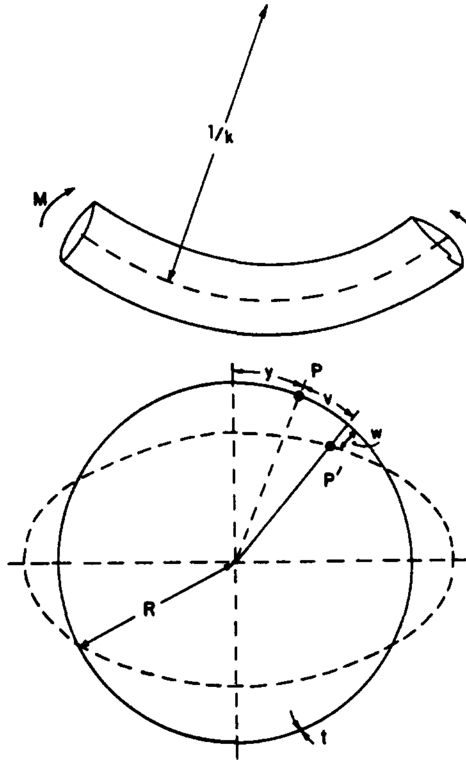


Fig. 1.

Furthermore, as will be discussed below, the shell middle surface is taken to be inextensional in the circumferential direction, so that

$$\eta = 0. \tag{13}$$

This imposes a constraint on $v$ and $w$. Variations of the two displacements are related by

$$\delta\eta = (1+e)\delta e + \beta\delta\beta = 0. \tag{14}$$

A finite rotation $\phi$ is defined by

$$\sin\phi = \beta. \tag{15}$$

The circumferential bending strain is

$$K_{yy} = \frac{1}{\sqrt{1-\beta^2}}\frac{d\beta}{dy}. \tag{16}$$

In this study it will not be assumed that $\eta \approx e$ and $\phi \approx \beta$ (as Brazier assumed), but that the nonlinear forms will be retained. The total axial and circumferential strains can be represented as

$$\epsilon_{xx} = E_{xx} + zK_{xx}; \qquad \epsilon_{yy} = E_{yy} + zK_{yy} \tag{17}$$

where $K_{xx}$ is the axial bending strain and the relation between $E_{yy}$ and $\eta$ will be discussed below.

## CONSTITUTIVE RELATIONS

A Ramberg–Osgood[5] uniaxial stress–strain curve is adopted to characterize the material, i.e.

$$\epsilon = \frac{\sigma}{E}\left[1 + \frac{3}{7}\left|\frac{\sigma}{\sigma_Y}\right|^{n-1}\right] \tag{18}$$

where $E$ is Young's modulus and $\sigma_Y$ is the effective yield stress. The $J_2$ deformation theory of plasticity will be used to generalize (18) to multi-axial states. In the prebuckling ovalization problem, plastic deformation is expected to be nearly proportional over most of the shell. Thus, little difference between results based on $J_2$ deformation theory and those on the corresponding incremental theory, $J_2$ flow theory, are expected. For the bifurcation calculation, $J_2$ deformation theory is again used since in this problem, as in the cylindrical shell under axial compression as well as in other plastic buckling problems, $J_2$ flow theory leads to unrealistically high bifurcation predictions[6, 7]. Finite strain effects are not taken into account in this study. This is consistent with the fact that the stress levels at buckling are small compared to the instantaneous moduli. The constitutive law is a small strain relation. One of the fortunate sidelights of using deformation theory, which will be exploited in the present study, is that the solution need not be obtained incrementally.

With non-zero stresses $\sigma_{xx}$ and $\sigma_{yy}$, the two-dimensional constitutive relations are

$$\epsilon_{xx} = \frac{1}{E_s}(\sigma_{xx} - \nu_s\sigma_{yy}); \qquad \epsilon_{yy} = \frac{1}{E_s}(\sigma_{yy} - \nu_s\sigma_{xx}). \tag{19}$$

The secant modulus $E_s$ is

$$\frac{1}{E_s} = \frac{1}{E}\left[1 + \frac{3}{7}\left(\frac{\sigma_e}{\sigma_Y}\right)^{n-1}\right] \tag{20}$$

where the effective stress is

$$\sigma_e = (\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy})^{1/2} \tag{21}$$

and the parameter $\nu_s$ is given in terms of $\nu$, Poisson's ratio, as

$$\nu_s = \frac{1}{2} + \frac{E_s}{E}\left(\nu - \frac{1}{2}\right). \tag{22}$$

## VARIATIONAL EQUATION OF EQUILIBRIUM

The overall curvature $\kappa$ is taken as the prescribed loading quantity in the calculations which will be described in the next section. The variation in the internal virtual work per unit length of the shell is

$$\delta(IVW) = \int_0^{2\pi R}\int_{-t/2}^{t/2}(\sigma_{xx}\delta\epsilon_{xx} + \sigma_{yy}\delta\epsilon_{yy})\,\mathrm{d}z\,\mathrm{d}y. \tag{23}$$

With resultant stress and moment quantities defined as

$$(N_{xx}, N_{yy}) = \int_{-t/2}^{t/2}(\sigma_{xx}, \sigma_{yy})\,\mathrm{d}z \tag{24}$$

$$(M_{xx}, M_{yy}) = \int_{-t/2}^{t/2}(\sigma_{xx}, \sigma_{yy})\,z\,\mathrm{d}z \tag{25}$$

then, with use of (17), the expression (23) becomes

$$\delta(IVW) = \int_0^{2\pi R}(M_{xx}\delta K_{xx} + M_{yy}\delta K_{yy} + N_{xx}\delta E_{xx} + N_{yy}\delta E_{yy})\,\mathrm{d}y. \tag{26}$$

The most important terms in (26) are the terms associated with ovalization, $M_{yy}\delta K_{yy}$, and the term due to axial stretch or compression from overall bending, $N_{xx}\delta E_{xx}$. These are the terms which enter into Brazier's solution. For an unpressurized cylindrical shell in bending $N_{yy}$ is zero at the fibers of extreme tension and compression and generally will be very small compared to the maximum value of $N_{xx}$. In the elastic range it can be shown from Brazier's solution that $N_{yy} = O(tN_{xx}/R)$. Thus, this term will be neglected in (26), and in the constitutive law. It will be required that the stress $\sigma_{yy}$ satisfy (approximately)

$$N_{yy} = \int_{-t/2}^{t/2} \sigma_{yy}\, dz = 0. \tag{27}$$

The bending contribution $M_{xx}\delta K_{xx}$ will also be neglected in the internal virtual work. The work done by $M_{xx}$ is always small compared to the work done by $N_{xx}$ and $M_{yy}$. For example, for the elastic problem at the maximum moment, $M_{xx}K_{xx} = O(t/R)\cdot M_{yy}K_{yy}$. Thus, with $\kappa$ prescribed the variational statement of equilibrium used here is

$$\int_0^{2\pi R} (M_{yy}\delta K_{yy} + N_{xx}\delta E_{xx})\, dy = 0. \tag{28}$$

## SOLUTION METHOD

The in-plane displacements $v$ and $w$ are assumed to be four-way symmetrical and are written in the form

$$w = R\sum_{i=0}^{N} a_i \cos(2iy/R), \qquad v = R\sum_{i=1}^{N} b_i \sin(2iy/R) \tag{29}$$

where the number of terms $N$ will be chosen to ensure satisfactory convergence. This assumption is appropriate for the present formulation for several reasons. Firstly, the longitudinal stretching strain $E_{xx}$ as developed in (8) and (9) does not have any quadratic terms in it to lead to possible non-symmetric forms for the tension and compression sides of the shell. Secondly, the hoop strain $\eta$ is set to zero around the middle surface of the shell, and thus the quadratic terms associated with $\eta$ do not necessarily dictate a change in form from the compression side to the tension side of the shell. Thirdly, it can be shown that this assumption insures that the net force in the longitudinal direction is zero. A result of the assumption is that both terms in the variational statement of equilibrium (28) are even about $y = \pi R/2$, as well as $y = 0$, and thus only one-quarter of the shell need be used for analysis.

The system of equations used in solving for the maximum moment are highly nonlinear, even in the elastic case; therefore, it was necessary to use a numerical Newton–Raphson procedure similar to that used in [8].

By using the expansions for $w$ and $v$ (29) in (10)–(13), the condition for inextensionality can be written in the form

$$\sum_{i=0}^{2N} C_i \cos(2iy/R) = 0 \tag{30}$$

where the $C_i$'s are functions of the $a_i$'s and $b_i$'s. The expansion (30) is truncated to only $N+1$ terms, that is

$$\sum_{i=0}^{N} C_i \cos(2iy/R) = 0. \tag{31}$$

Because of the orthogonality of the functions $\cos(2iy/R)$, the following equations may be written

$$C_i = 0 \qquad i = 0, 1, \ldots, N. \tag{32}$$

The $C_i$'s are calculated to be

$$C_0 = \frac{a_0^2}{2} + a_0 + \frac{1}{4}\sum_{j=1}^{N} [(1+4j^2)(a_j^2 + b_j^2) + 8a_jb_j]$$

$$C_1 = (1+a_0)(2b_1+a_1)+\frac{1}{2}\sum_{j=1}^{N-1}[(4j+4j^2+1)(b_jb_{j+1}+a_ja_{j+1})$$
$$+2(1+2j)(a_jb_{j+1}+a_{j+1}b_j)]$$
$$C_i = (1+a_0)(2ib_i+a_i)+\frac{1}{2}\sum_{j=1}^{N-i}[(4ij+4j^2+1)(b_jb_{i+j}+a_ja_{i+j})$$
$$+2(i+2j)(a_jb_{i+j}+a_{i+j}b_j)]+\frac{1}{4}\sum_{j=1}^{i-1}[(4ij-4j^2-1)(b_jb_{i-j}-a_ja_{i-j})$$
$$+2(i-2j)(a_jb_{i-j}-a_{i-j}b_j)] \qquad (i=2,3,\ldots,N-1)$$
$$C_N = (1+a_0)(2Nb_N+a_N)+\frac{1}{4}\sum_{j=1}^{N-1}[(4Nj-4j^2-1)(b_jb_{N-j}-a_ja_{N-j})$$
$$+2(N-2j)(a_jb_{N-j}-a_{N-j}b_j)]. \tag{33}$$

These equations determine $N+1$ of the coefficients in terms of the other $N$. Following the same procedures to obtain a form for (14) yields

$$\delta C_i = 0 \qquad (i=0,1,\ldots,N) \tag{34}$$

where the variations are taken with respect to the $a_i$'s and $b_i$'s. These are linear equations relating the variations of the coefficients. There are $N+1$ of these equations as well. If it is assumed that $a_1, a_2, \ldots, a_N$ are the "independent" variables, and $b_1, b_2, \ldots, b_N$ and $a_0$ are the "dependent" ones, then eqns (33) provide the dependent variables in terms of the independent ones, and eqns (34) can be summarized as

$$\delta a_0 = \sum_{j=1}^{N}\gamma_{0j}\delta a_j; \qquad \delta b_i = \sum_{j=1}^{N}\gamma_{ij}\delta a_j, \qquad i=1,2,\ldots,N. \tag{35}$$

The coefficients $\gamma_{ij}$ are functions of the $a_i$'s and $b_i$'s.

Equations (33) were solved by a Newton's method. For a given set of $a_i$'s, a good initial guess is $b_i = -a_i/2i$ and $a_0 = 0$. This is exact if the linearization $\eta = e$ is made, as Brazier did.

With the $a_i$'s and $b_i$'s known, the nonzero strains $E_{xx}$ and $K_{yy}$ are readily obtained. These, along with a choice for $E_{yy}$, are used to calculate the stresses from the constitutive relations. Below are two approximate methods:

*Method A*

It is first assumed that $\sigma_{yy} = 0$ on the middle surface of the shell. The axial stress on the middle surface of the shell is calculated from

$$E_{xx} = \sigma_{xx}/E_s. \tag{36}$$

It is then assumed that the moduli are constant throughout the thickness of the shell. The strain $E_{yy}$ is chosen to be

$$E_{yy} = -\nu_s E_{xx}. \tag{37}$$

The results of these assumptions are that $N_{xx}$ and $M_{yy}$ are readily calculable and the condition on $N_{yy}$ (27) is satisfied; however, $\sigma_{yy}$ is on the same order as $\sigma_{xx}$ at the inner and outer fibers of the shell, and thus the moduli are incorrect.

*Method B*

Again it is assumed that $\sigma_{yy} = 0$ on the middle surface of the shell, and (36) and (37) are employed again. This time, the change in moduli through the thickness is taken into account; and while (27) is not satisfied, it was found that $|N_{yy}|^{\max}/|N_{yy}|^{\max} < 0.01$ for all cases studied.

When the amount of plastic deformation is moderate, the difference in results between the two methods is small; however, a greater difference was found when there was large plastic deformation. Method B was used in this study because the error in not satisfying (27) was far below the error in using the incorrect moduli of Method A, where it is noted that this error would increase with increasing plastic deformation.

A more accurate method would employ a Newton's method with method B providing an initial guess for $E_{yy}$ around the middle surface of the shell. It should be noted that for an elastic shell, both Method A and Method B are equivalent and exact.

Equations (8), (9), (16) and (35) are combined into (28) to yield the variational equation in the form

$$\sum_{i=1}^{N} \phi_i \delta a_i = 0 \tag{38}$$

which yield the $N$ field equations

$$\phi_i = 0 \qquad (i = 1, 2, \ldots, N) \tag{39}$$

where the $\phi_i$ are calculated to be

$$\begin{aligned}\phi_i = \int_0^{2\pi R} \Bigg\{ \frac{M_{yy}}{R} \Bigg[ &\frac{1}{(1-\beta^2)^{1/2}} \left( \sum_{j=1}^{N} 2j\gamma_{ji} \cos\frac{2jy}{R} + 4i^2 \cos\frac{2iy}{R} \right) \\ &+ \frac{\beta}{(1-\beta^2)^{3/2}} \left( \sum_{j=1}^{N} \gamma_{ji} \sin\frac{2jy}{R} + 2i \sin\frac{2iy}{R} \right) \Bigg] \\ &- N_{xx} R\kappa \left[ \cos\frac{y}{R} \left( \gamma_{0i} + \cos\frac{2iy}{R} \right) - \sin\frac{y}{R} \sum_{j=1}^{N} \gamma_{ji} \sin\frac{2jy}{R} \right] \Bigg\} \mathrm{d}y \qquad i = 1, 2, \ldots, N.\end{aligned} \tag{40}$$

These equations may be used to solve for the $a_i$'s, $i = 1, 2, \ldots, N$, as a function of $\kappa$ using, again, a Newton's method. Every time one or all of $a_i$'s are changed, it is necessary to recalculate the $b_i$'s, strains, stresses, and so on. Finally, the curvature $\kappa$ is changed so that the applied moment as calculated by

$$M = \int_0^{2\pi R} N_{xx} h \, \mathrm{d}y \tag{41}$$

is maximized. The subscript or superscript $\mu$ will denote quantities associated with this limit state.

## RESULTS

In the actual numerical procedure used, it was found that the choice $N = 4$ in the expansions of $v$ and $w$ was sufficient to give accurate results. Table 1 is a comparison of the elastic results for this calculation with Brazier's[2] calculation for different numbers of terms $N$ in the expansions. The calculated values of the quantities presented change by less than 1% when the number of terms increases from 2 to 4. This demonstrates the strong convergence properties of the method of calculation. The greatest discrepancy between the results occurs between the values for the maximum inward deflection. Furthermore, the results of this study agree closely with the limit states found by Fabian[3], and those of Akselrad[11] and Almroth and Starnes[12] in the limit as the slenderness ratio goes to infinity.

An important nondimensional parameter is

$$\frac{\sigma_Y}{\sigma_c^e} = [3(1-\nu^2)]^{1/2} \frac{R}{t} \frac{\sigma_Y}{E} \tag{42}$$

which is the ratio of the effective yield stress of (18) to the bifurcation stress of the elastic, perfectly circular cylindrical shell in axial compression, as given in (1). Values of $R/t$ and $\sigma_Y/E$ enter only in combination through the parameter (42) and need not be specified separately. Figure 2 shows curves of $M_\mu$, the limit moment, normalized by $M_c^e$ as given in (2), against $\sigma_Y/\sigma_c^e$, for a material characterized by (18) with $n = 3, 5$ and 10 and for $\nu = 0.3$. Note that when $\sigma_Y/\sigma_c^e > 1$, the results are close to the elastic results, and when $\sigma_Y/\sigma_c^e \ll 1$, the results indicate a large reduction in the limit moment.

Table 1. Comparison of nonlinear and linear analyses for the collapse of a long elastic cylindrical shell in pure bending

| Number of terms in expansion (31), N | 2 | 3 | 4 | Ref. [2] |
|---|---|---|---|---|
| $\frac{2R\kappa_\mu^e}{\epsilon_c^e}$ | 1.6570 | 1.6520 | 1.6522 | 1.6330 |
| $\frac{M_\mu^e}{M_c^e}$ | .5297 | .5287 | .5286 | .5443 |
| $\frac{\tau_\mu^e}{\sigma_c^e}$ | .6144 | .6133 | .6134 | .6352 |
| $\frac{\delta_\mu^e}{R}$ | .2584 | .2574 | .2574 | .2222 |

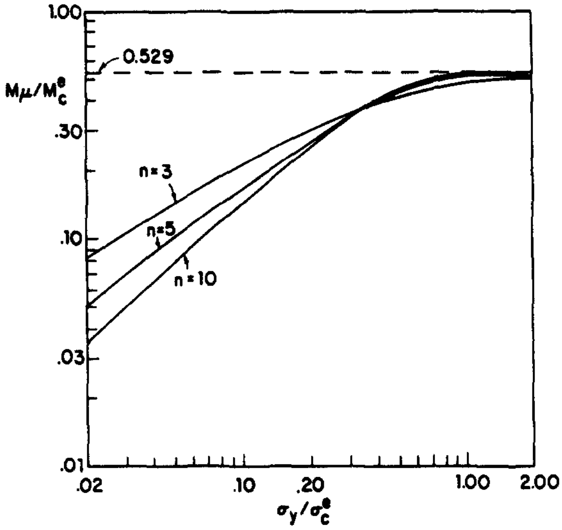


Fig. 2.

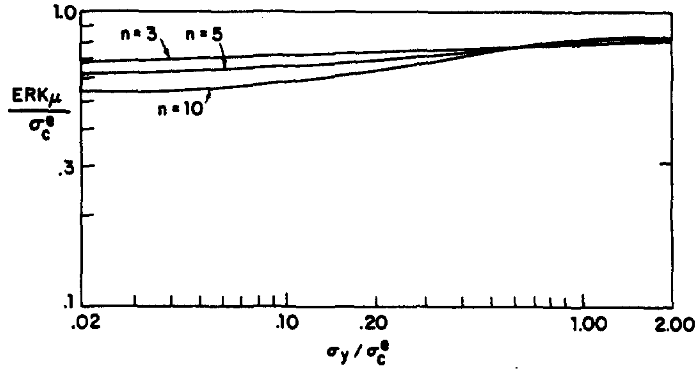


Fig. 3.

Figure 3 is a plot of $ER\kappa_\mu/\sigma_c^e$ against $\sigma_Y/\sigma_c^e$. The critical curvature is the parameter of greatest interest in connection with the laying of undersea pipelines[9]. This parameter appears to be relatively insensitive to the amount of plastic deformation as measured by $\sigma_Y/\sigma_c^e$, with only a small decrease in value with increasing plasticity.

The maximum inward deflection at maximum load, $\delta_\mu$, decreases with increasing plasticity, from the elastic value of approximately $0.26R$ to about $0.5R$ for the cases studied. The maximum inward deflection was always found to be greater than the maximum outward deflection, that is, the parameter $a_0$ is always negative. This result is expected since $a_0$ is very likely the dominant term in $e$, which, from (13), must be negative. Moment curvature relationships for loads below that of the limit state were calculated in a few cases and found to be comparable with the results of Ades[4].

Figure 4 is a plot from Reddy[10] of the outer fiber strain associated with the limit moment from tests performed by Reddy and other investigators where the two lower curves shown in the figure represent the empirical results. The constant of proportionality relating this strain and $t/R$ decreases with reduced strain hardening. This constant $\alpha$ can be derived from quantities used in the present study from

$$\alpha = [3(1-\nu^2)]^{1/2}\left[1-\frac{\delta_\mu}{R}\right]\frac{\kappa_\mu}{\kappa_c^e} \tag{43}$$

where $\kappa_c^e$ is given by (5). For smaller values of $\sigma_Y/\sigma_c^e$, which correspond to the range of $R/t$ shown in Fig. 4, the value of $\alpha$ is nearly constant for all three exponents $n$. These values are 0.37, 0.35 and 0.30 for $n = 3$, 5 and 10 respectively, with $\nu = 0.3$, thus providing excellent agreement with the trend of the experimental results in Reddy's plot. For much thinner shells, $\alpha$ approached the elastic result of 0.3707 for all three exponents.

Bifurcation from this nonlinear state is possible at a load less than the limit moment. This was investigated in a manner similar to the elastic study of Fabian[3]. Table 2 is a summary of selected results, the subscript $b$ being used to designate quantities associated with the bifurcated state. The moment at bifurcation was not reduced more than a few percent below the maximum moment $M_\mu$ for the cases studied, though the curvature was reduced between 17 and

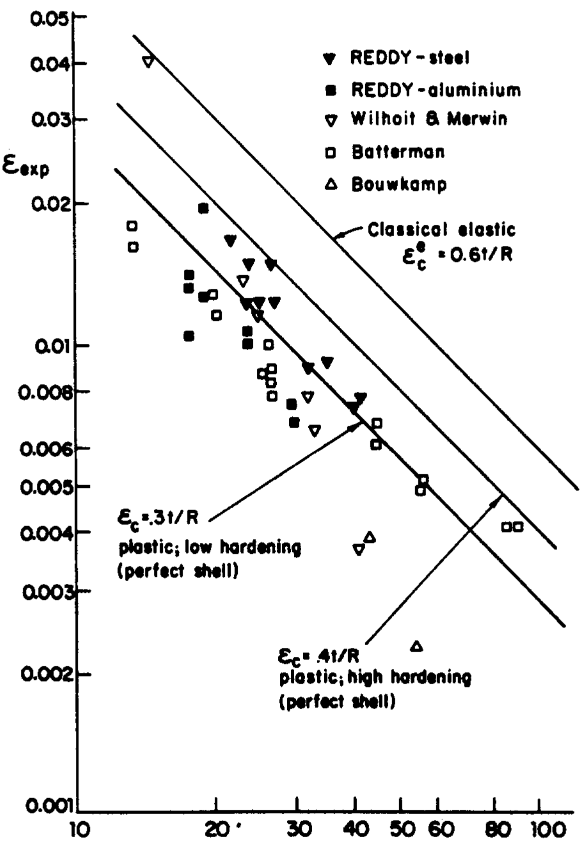


Fig. 4.

Table 2. Comparison of collapse modes for long cylindrical shells in pure bending for $[3(1-\nu^2)]^{1/2}\frac{\sigma_Y}{E} = 10^{-3}$.

| R/t | 30 | 50 | 100 |
|---|---|---|---|
| | a) n=3 | | |
| $\frac{\kappa_b}{\kappa_\mu}$ | .83 | .80 | .78 |
| $\frac{\tau_b}{\tau_\mu}$ | .95 | .94 | .92 |
| $\frac{M_b}{M_\mu}$ | .983 | .976 | .965 |
| | b) n=5 | | |
| $\frac{\kappa_b}{\kappa_\mu}$ | .83 | .795 | .755 |
| $\frac{\tau_b}{\tau_\mu}$ | .967 | .958 | .942 |
| $\frac{M_b}{M_\mu}$ | .990 | .985 | .973 |
| | c) n=10 | | |
| $\frac{\kappa_b}{\kappa_\mu}$ | .815 | .775 | .725 |
| $\frac{\tau_b}{\tau_\mu}$ | .981 | .975 | .962 |
| $\frac{M_b}{M_\mu}$ | .996 | .992 | .982 |
| | d) Elastic case: R/t=50 | | |
| $\frac{\kappa_b^e}{\kappa_\mu^e} = .75$ | $\frac{\tau_b^e}{\tau_\mu^e} = .86$ | $\frac{M_b^e}{M_\mu^e} = .93$ | |

28% below $\kappa_\mu$. In addition, an elastic result is presented to demonstrate agreement with Fabian. It should also be noted that $R/t$ and $\sigma_Y/E$ must be specified separately for the bifurcation problem.

The effect of initial imperfections were not taken into account in this study, though it is noted that for thin shells (and thus higher values of $\sigma_Y/\sigma_c^e$, resulting in nearly elastic behavior) a significant reduction in load capacity due to initial imperfections is expected; however for thicker shells (and thus lower values of $\sigma_Y/\sigma_c^e$ resulting in large plastic deformation) the effect of the imperfections may not be quite so important. The combined effects of initial imperfections and bifurcation can probably account for the results of Palmer[9] for large undersea pipelines.

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